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Introduction

The Gemara in Succos 8b and Eruvin 76b both quote a *ma'amar* from Rabbi Yochanan that a 4 by 4 square can be circumscribed by a circle that has an *hakayfoh* of 24. The Gemara then brings proof from the Rabbis of Caesarea who state, "a circle inscribed in a square, is one-fourth; a square which is inscribed in a circle, is one-half". The normal meaning of *hakayfoh* is circumference or perimeter. This leads to a large difference between the 24 of R' Yochanan and the 16.8 the Gemara calculates (or even the 17.77 with more accurate calculations).

A diagram of the circle and squares is presented with a table that identifies the geometric items being referenced by the Gemara and commentaries. Each line of the table is numbered and explained in detail below. Then follows a discussion of the Gemara according to various commentaries, identifying which item they are referring to, followed by a conclusion that proposes a solution that accepts the words of Rabbi Yochanan and the Rabbis of Caesarea as correct by showing the results when their words are explained as referring to area instead of circumference.

The diagram and table should be easy to follow as they use only two squares and a circle and use only elementary algebra and geometry. All terms are defined.

Acknowledgements

My interest in these two Gemaras was aroused by a comment by Dr. Feldman in [Feldman] pg. 30. Further research found Rabbi Mordecai Kornfeld [Kornfeld] who explained the issues and pointed the way to further research. The actual learning of the sources, and providing many of them, would not have been accomplished without the help and guidance of Rabbi Chaim Kastel. Finally I wish to thank my wife Shirley, without whose patience and understanding this could not, and would not, have been done.

I would also like to thank Menachem Epstein, R' Yitzchok Ginsberg, and Dr. Richard Mosak who were kind enough to review early drafts, but I remain responsible for any mistakes.

Knowing that there probably are mistakes I would greatly appreciate anyone who finds any errors or suggestions for improvement, please email me at abraham@walfish.com. In particular, the G'ra in Eruvin has an explanation of the issue but the text is not certain and I have not been able to understand how it explains the difficulties.

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Description of the diagram and table

The Gemara in Succos 8b and Eruvin 76b discusses a circle circumscribing a square and inscribed in a square. The following diagram and table describes the circle O with an inscribed square *ABCD* and a circumscribed square EFGH. The various lines (rows) of the table describe the geometric items and calculations referred to by the Gemara and various commentaries.

The **Item** identifies the geometric figure described. The **Calculated approximation** describes the value using modern mathematics. The **Talmudic approximation** describes the value using the approximation used by the Gemara or the various commentaries. Chazal knew their approximations could be more accurate but deemed them close enough for practical Halacha. Only when the differences were great did the Gemara object (see Appendix B).

Definitions and Formulas

+ – Plus – Addition; 3 + 4 = 7

– – Minus – Subtraction ; 4 – 3 = 1

* – Times – Multiplication; 3 * 4 = 12

/ – Divide – Division; 12 / 4 = 3

~ – Approximately

( ) – Parentheses; grouping of terms. The items in the parentheses are evaluated first. Thus:
(2 + 3) * 4 = 20; 2 + (3 * 4) = 14

Square – *Rebuia* – a four sided figure where each side is the same length and each corner is a right (90 degree) angle.

Perimeter of a square – *hakayfoh* – the length of the outside of the square, the sum of the four sides.

Area – *Shibur* or *Tishbores* (term used by the Me'iri), *Shetach* (modern Hebrew) – the area of a figure measures the size of the region enclosed by the figure. This is usually expressed in terms of some square unit. A few examples of the units used are square feet, square centimeters, square inches, square kilometers, square tefachim, or square amos.

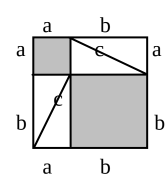
Area of a square – the multiplication of the 2 sides.

Right Triangle – a three-sided figure where two sides are perpendicular, i.e. they meet at a right (90° [degree]) angle. The hypotenuse is the side opposite the right angle. The sides are conventionally named a, b, and c, where c is the hypotenuse.

x^2 – Square of x; x * x.

Pythagorean Theorem – for any right triangle the sum of the squares of the sides is equal to the square of the hypotenuse or, for a conventionally named right triangle, $a^2 + b^2 = c^2$.

A beautiful dissection proof uses two figures (Gardner 1984, p. 154)[1]. Both figures have an area equal to $(a + b)^2$. Both have 4 abc right triangles. Subtract the 4 triangles from each figure. The top figure has $a^2 + b^2$ left and the bottom has c^2 left. Therefore $a^2 + b^2 = c^2$.



For an algebraic solution, you can use just the bottom figure where: the area of the large square is

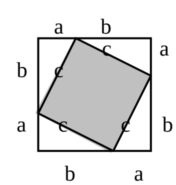
(1) $(a + b)^2 = (a + b) * (a + b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2$.

Also by looking at the shapes in the square there are 4 triangles that have a total area of 2ab (the 2 triangles on the diagonal when put together on side c form a rectangle ab) and the center square of c^2 so the area of the large square is

(2) $2ab + c^2$.

Subtracting 2ab from both views [from (1) and (2)] yields

(3) $a^2 + b^2 = c^2$.



[1] http://mathworld.wolfram.com/PythagoreanTheorem.html

$\sqrt{(x)}$ – Square root of x. The number that when multiplied by itself equals x. Thus y is the $\sqrt{(x)}$, means y * y = x; and $x = \sqrt{(x^2)}$ because $x * x = x^2$.

Diagonal of a square – *Alachsona* – is the distance from one corner of the square to the opposite corner and is equal to the length of a side times the square root of 2. This can be shown by using the Pythagorean Theorem for a square with side s. The diagonal, c, is the hypotenuse of a right triangle with a & b = s. So: $c^2 = s^2 + s^2 = s^2 * 2$. Take the square root of each side and $c = s * \sqrt{(2)}$.

The $\sqrt{(2)}$ is an irrational[2] number approximately equal to 1.4142135623731 accurate to 14 decimal digits. The closest approximation in a one-digit fraction is 5 divided into 7 (1 2/5) = 1.4, which is used by the Gemara and is accurate to 2 decimal digits. [Ga'aly Mesechos] and the Gra use 113 / 80 = 1.4125, which is accurate to 4 decimal digits.

π (Pi) – is the ratio of the circumference of a circle to its diameter. It is an irrational[3] number and cannot be expressed as an exact fraction. π is ~= 3.14159265358979 accurate to 15 decimal digits. The closest approximation in a one-digit fraction is 1 divided into 3 = 3, which is used by the Gemara and is accurate to 1 decimal digit. The closest approximation with a two-digit fraction is 7 divided into 22 ~= 3.142857, which is accurate to 3 decimal digits and is also used by Chazal[4]. The closest approximation with a three-digit fraction[5] is 113 divided into 355 ~= 3.141592920 which is accurate to 7 decimal digits.

Radius of a circle – r – the distance from the center of the circle to the outside of the circle.

Diameter of a circle – d – the longest distance across the circle, which is through the center and is 2 times the radius.

Circumference of a circle – *hakayfoh* – the length of the outside of the circle, which is π times the diameter; $\pi * d$ or $2 * \pi * r$.

Area of a circle – for a circle[6], π times the radius squared or $\pi * r^2$.

Square *ABCD*

The first 5 lines describe the inner square *ABCD*.

Length

1. Describes the length of each side of a square 4 by 4 units long[7].

Perimeter

2. Describes the perimeter of square *ABCD*. The sum of the length of its sides equals 16 units long.

Area

3. Describes the area of square *ABCD*. The product of side times side equals 16 square units[8]. In this particular case of a 4 x 4 square, the perimeter equals the area, which can cause some confusion.

4. Describes the area of square *ABCD* as a relationship between the square *ABCD* and the square EFGH. I.e. the area of *ABCD* is equal to the area of EFGH minus one half the area of EFGH. This is the reciprocal of line 25 and is obvious from noting that triangle ABO is congruent to triangle ABE and there are 4 such pairs.

5. Describes the area of square *ABCD* as a relationship between the square *ABCD* and the circle O. I.e. the area of *ABCD* is equal to the area of O minus one-third the area of O. This is the reciprocal of line 20 and works for the Talmudic approximation of $\pi = 3$.

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[2] An irrational number is any real number that is not a rational number, i.e., it is not of the form *a* divided by *b* where *a* and *b* are integers and *b* is not equal to zero.

[3] π is additionally a transcendental number, i.e. a complex number that is not algebraic, that is, not the solution of a non-zero polynomial equation with integer (or, equivalently, rational) coefficients.

[4] 49th Middoth of R. Nathan [Feldman pg 23]

[5] Note: this approximation uses the first three odd digits twice each.

[6] More properly a disk (the area enclosed in a circle).

[7] The unit is amah in Succos and tefach in Eruvin.

[8] That is the unit-less, cardinal number.

Circle O

Lines 6 through 21 describe the circle O.

Length

6. Describes the diameter of O, which equals the diagonal of *ABCD*, which equals the side of EFGH. The diagonal[9] of the square is the length of the side times the square root of 2 ($\sqrt{2}$ ~= 1.41421). Chazal used one and two-fifths (1.4) as the approximation of the diagonal of a one by one square[10].
7. Describes the diameter of O, which equals the diagonal of *ABCD* with a closer approximation of 113 /20 (5.65). This utilizes 1 33/80 = 1.4125 as the approximation of the diagonal of a 1 by 1 square[11].
8. Describes OL, the radius of O. One half of line 6 (the diameter of O or the diagonal of *ABCD*).
9. Describes OK, the part of radius OL inside the square *ABCD*, since the intersection of the diagonals of a square meet at the center of the square (and of the circumscribed circle).
10. Describes KL the part of radius OL outside the square *ABCD* but inside the circle O.

Circumference

11. Describes the circumference of O, equals π times the diameter.
12. Describes the circumference of O as a relationship between the circle O and the square EFGH. I.e. the circumference of O is equal to the perimeter of EFGH minus one-fourth the perimeter of EFGH. This is the reciprocal of line 23 and works for the Talmudic approximation of $\pi = 3$.
13. Describes the length of the arc LB, which is the circumference divided by eight. Rashi and the Gra use this.

Area

14. Describes the area of circle O, π times the radius squared. Since r = the square root of 8 (by line 8), the area = π times 8 ~= 25.13 or 24 by Talmudic approximation.
15. Describes the area of circle O with values $\pi = 3$ and the radius = 2.8 (line 8 = line 6 divided by 2).
16. Describes the area of circle O with values $\pi = 3$ and the radius = 113 / 20 / 2 (line 7 divided by 2).
17. Describes the area of circle O with values $\pi = 22 / 7$ and the radius = 2.8 (line 8 = line 6 divided by 2).
18. Describes the area of circle O with values $\pi = 22 / 7$ and the radius = 113 / 20 / 2 (line 7 divided by 2).
19. Describes the area of circle O as a relationship between the circle O and the square EFGH. I.e. the area of O is equal to the area of EFGH minus one-fourth the area of EFGH. This is the reciprocal of line 26 and works for the Talmudic approximation of $\pi = 3$.
20. Describes the area of circle O as a relationship between the circle O and the square *ABCD*. I.e. the area of O is equal to the area of *ABCD* plus one-half the area of *ABCD*. This is the reciprocal of line 5 and works for the Talmudic approximation of $\pi = 3$.
21. Describes the area of the segment CBL of circle O. It is the area of O minus the area of *ABCD*, divided by 4, as there are 4 such segments.

Square EFGH

Lines 22 through 26 describe the square EFGH.

Perimeter

22. Describes the perimeter of square EFGH. The sum of the length of it's sides. Equals 4 times the square root of 32 (line 6).

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[9] This is obvious from the Pythagorean Theorem; $a^2 * b^2 = c^2$ where a and b are the two sides of a right triangle and c is the hypotenuse (see the definition of Diagonal above) where a = line AB, b = line BC, and c =line AC.

[10] Eruvin 57a

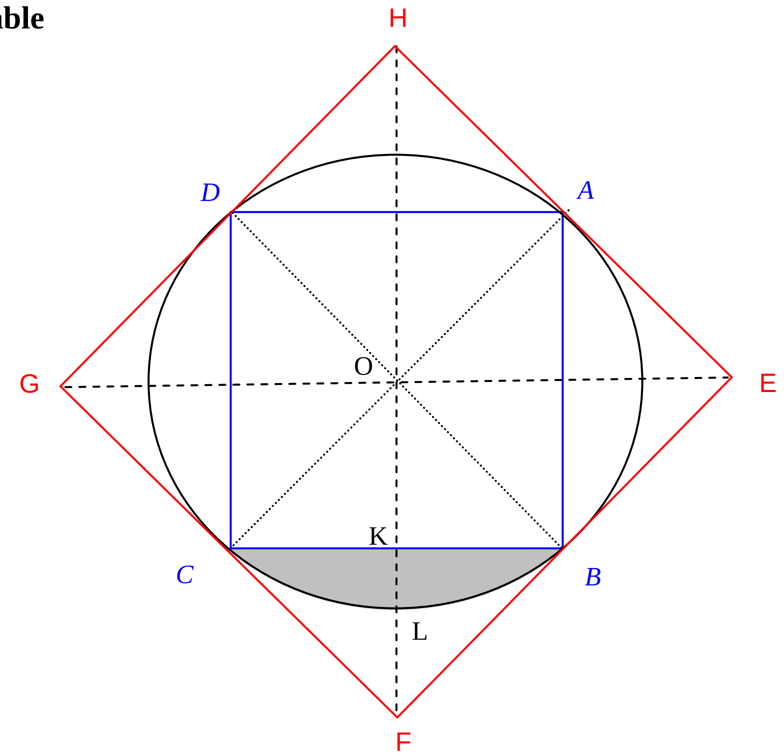
[11] Based on [Ga'alay Masechos] and the Gra in Eruvin 76b

23. Describes the perimeter of square EFGH as a relationship between the square EFGH and the circle O. I.e. the perimeter of EFGH is equal to the circumference of O plus one-third the circumference of O. This is the reciprocal of line 12 and works for the Talmudic approximation of $\pi = 3$.

Area

24. Describes the area of square EFGH. The product of side times side. Since the side of EFGH equals the square root of 32 (line 6) the area equals 32. Note: if we use the Talmudic approximation from lines 6 or 7, we would get 31.36 or 31.9225, but by line 25, Chazal knew the area is equal to 32.
25. Describes the area of square EFGH as a relationship between the square EFGH and the square *ABCD*. I.e. the area of EFGH is equal to the area of *ABCD* plus the area of *ABCD*. This is the reciprocal of line 4 and is obvious from noting that triangle ABO is congruent to triangle ABE and there are 4 such pairs.
26. Describes the area of square EFGH as a relationship between the square EFGH and the circle O. I.e. the area of EFGH is equal to the area of O plus one-third the area of O. This is the reciprocal of line 19 and works for the Talmudic approximation of $\pi = 3$

Diagram and Table



	Item	**Calculated approximation**	**Talmudic approximation**
1.	*AB = BC = CD = DA*	*4*	*4*
2.	*Perimeter of ABCD*	*4 + 4 + 4 + 4 = 16*	*4 + 4 + 4 + 4 =* ***16***
3.	*Area of ABCD =*	*4 * 4 = 16*	*4 * 4 =* ***16***
4.	*(Area of* EFGH*) * (1 – ½)*	*32 – 32/2 = 32 – 16 = 16*	*32 – 32/2 = 32 – 16 = 16*
5.	*(Area of* O*) * (1 – 1/3)*		*24 – 24/3 = 24 – 8 = 16*
6.	*AC* = d = EF = √(AB^2+BC^2)	4*√(2) = √(16*2) =√(32) = 5.657	4 * (1 2/5) = 5 3/5 = **5.6**
7.	AC		4*(1 33/80) = 113/20 = 5.65
8.	Radius of O = OL = ½ d = ½ *AC*	√(32)/2 = √(32)/√4 = √(8) = 2.83	(4 * (1 2/5)) / 2 = 2 4/5 = 2.8
9.	OK = ½ *AB*	2	2
10.	KL = OL – OK	.83	4 / 5 = .8
11.	Circumference of O = π*d =	π * √(32) = 17.77	3 * (5 3/5) = 16 4/5 = 17–1/5 = **16.8**
12.	(Perimeter of EFGH)*(1 – ¼)		4*(5 3/5)–(5 3/5) = 3*(5 3/5) = 16.8
13.	Arc LB = π*d / 8	17.77 / 8 = 2.22	16.8 / 8 = 2.1
14.	Area of O = $\pi * r^2$ =	$\pi * (\sqrt{(8)})^2 = \pi * 8$ = 25.13	3 * 8 = **24**
15.	π=3, r=2.8		23.52
16.	π=3, r=113/20/2=2.825		23.94
17.	π=22/7, r=2.8		24.64
18.	π=22/7, r=113/20/2=2.825		25.08
19.	(Area of EFGH) * (1 – ¼)		32 – 32/4 = 32 – 8 = **24**
20.	(Area of *ABCD*) * (1 + ½)		16 + 16/2 = 16 + 8 = **24**
21.	Area of segment CBL	(π * 8 – 16) / 4 = 2.28	(24 – 16) / 4 = **2**
22.	Perimeter of EFGH =	4*√(32) = 22.6	4 * (5 3/5) = **22.4**
23.	(Circumference of O) * (1 + 1/3)		3*(5 3/5)+(5 3/5) = 4*(5 3/5) = 22.4
24.	Area of EFGH =	√(32) * √(32) = 32	
25.	(Area of *ABCD*) * (1 + 1)	16 + 16 = 16 * 2 = 32	16 + 16 = 16 * 2 = **32**
26.	(Area of O) * (1 + 1/3)		24 + 24/3 = 24 + 8 = 32

Gemara

Succos 8a,b

Rabbi Yochanan says a round Sukah is Kosher if its *hakayfoh* is large enough to contain 24 people based on each person taking up one amah. The Gemara assumes circumference and a linear amah and demonstrates this can not be correct and makes adjustments to make it approximately correct. The Gemara then quotes the Rabbis of Caesarea in support of Rabbi Yochanan and says they are wrong.

Eruvin 76b

Rabbi Yochanan says a round *pesech* (opening) between two courtyards is valid to join them if it is 24 tefachim *b'hakayfoh* and a little more than two tefachim must be below 10 tefachim from the ground. As in Succos the Gemara assumes circumference, concludes the calculation is not correct for linear measure, and then quotes the Rabbis of Caesarea in support of Rabbi Yochanan.

Commentaries

Rashi

The following explanation of Rashi and Tosafos is taken verbatim from [Kornfeld].

> The Gemara comments that we can see that the circle around a square is not as large as the Rabbis of Caesarea posit. Based on the comments of Rashi elsewhere, though, we might suggest that Gemara is commenting only about the mathematical correctness of their statement; however, when considering the actual halachic applications, we do take into account their formula. In fact, we find in Eruvin (76a) that Rashi seems to have no difficulty with the statements of the Rabbis of Caesarea and Rabbi Yochanan. Perhaps Rashi held that the Rabbis of Caesarea were proposing a Halachic stringency: when determining a value (such as the circumference of a circle) by using the diagonal of a square for the purpose of a practical application in Halachah, we consider the diagonal to be equal to the sum of the two sides of the square or rectangle between the ends of the diagonal (since the lines of those two sides go from one end of the diagonal to the other). The reason for this is to prevent people from confusing the diagonal and the sum of two sides. In addition, physical reality does not permit for the application of puristic mathematics (for one reason, the actual diagonal of a square is the length of the side times the square root of two, which is an irrational number; second, it is not possible to draw a perfectly exact line or angle in the physical reality), and therefore the figure given as the diagonal of a square for purposes of determining Halachic applications (such as the size of a circular Sukah around that square) must take into consideration the largest possible diagonal of the right angle, which is the sum of the two sides. (Thus, if the sides of inscribed square are each 4 Tefachim, then the diagonal is viewed to be *8* Tefachim. The circle around that square, then, must have a diameter of 8 Tefachim, which means that its circumference must be *24* Tefachim, and not 16.8 which is what it would be based on the *actual* diameter of the square.)
>
> It could be that Rashi is consistent with his opinion elsewhere (Shabbos 85a, Eruvin 5a, 78a, 94b), where Rashi seems to count the diagonal of a rectangle as the sum of the two sides between the two ends of the diagonal. Rashi may hold that such a halachic definition is applied and may be relied upon entirely, both as a leniency and a stringency, with regard to Rabbinic rulings.

Tosafos

> Tosafos ([Succos] 8b, DH Rivu'a; Eruvin 76b, DH v'Rebbi Yochanan) suggests that the Rabbis of Caesarea were not giving the relationship of the *perimeter* of the inner square to the *circle* around it. Rather, they were giving the relationship of the *area* of the inner square to an *outer square* that is drawn around the circle which encloses the inner square. This is what they meant by saying that "when a circle is drawn around the outside of a square, the outer one's (i.e., the outer *square's*) perimeter is 50% larger than the inner one's." (See the second picture printed in Tosafos.)
>
> According to Tosafos, Rabbi Yochanan (both here and in Eruvin 86a) misunderstood the Rabbis of Caesarea.

Ritva

The Ritva concludes that the Rabbis of Caesarea are correct and the word "*v'lo hi*" in the Gemara in Succos were mistakenly added by the Rabonan Savora'i and should be deleted. [Ritva]. See Appendix A, Eruvin (pg. 14).

Me'iri

The Me'iri[12] in Eruvin 76, who cites it from the Ba'al ha'Me'or who consulted with Rabbi Yehuda Tibon, says that Rabbi Yochanan and the Rabbis of Caesarea could not have made such a large error in calculation and conclude they were talking about **area**, not perimeter or circumference and they are both right as we see in lines 19 and 20. Thus, the Gemara's calculation based on circumference is correct and Rabbi Yochanan and the Rabbis of Caesarea's calculation based on area is correct. This solution can be traced further back to a responsum of the Raavad in Temim Daim #223 [Tamim Daim], which is published with T'shuvos Rif. An Acharon, T'shuvos Galya Maseches #3 [Ga'aly Mesechos], offers this solution as well.

This explains Eruvin, in Succos however, the Gemara says "*v'lo he*". The Me'iri then explains that the calculation of Rabbi Yochanan and the Rabbis of Caesarea is based on π being equal to 3, and the Gemara knows that 3 is inaccurate. He goes on to use π equal to 22/7 to show the difference. He uses an example of a 7 by 7 EFGH square and doubles the numbers so he can eliminate fractions to illustrate the arithmetic. The table below uses the previous diagram and item numbers based on a side of EFGH = 7 and π = 22/7. The resulting ratio of the areas is 14, 11, and 7 not 16, 12, and 8 of Rabbi Yochanan and the Rabbis of Caesarea (see lines 25, 14, and 3 of the above table, divided by 2) and are thus not accurate.

Since the side of EFGH = 7 is given, the table below should be considered in reverse order, first the square EFGH (lines 22,24), then the circle O (lines 6,8,11,14), then the square *ABCD* (lines 1,2,4).

	Item	Calculated approximation	Half Tefachim (double the numbers)	Ratio
1.	*Side of ABCD*	*[see calculation below[13]] 7 / √2 = 4.95*		
2.	*Perimeter of ABCD*	*4 * (7 / √2) = 19.80*		
3.	***Area of ABCD***	***(7 / √2) * (7 / √2) = 7^2/2 = 49/2 = 24.5***	***(49/2)* 2 = 49***	***49/7 = 7***
6.	*AC* = d = EF	7		
8.	Radius of O=OL = ½d = ½ *AC*	7/2		
11.	Circumference of O = π*d	(22/7) * 7 = 22		
14.	**Area of O = π*r^2**	**22/7 * $(7/2)^2$ = 22/7 * 7/2 * 7/2 = 77/2**	**(77/2) * 2 = 77**	**77/7 = 11**
22.	Perimeter of EFGH	7 + 7 + 7 + 7 = 28		
24.	**Area of EFGH**	**7 * 7 = 49**	**49 * 2 = 98**	**98/7 = 14**

The Me'iri uses the word "Shibur" or "Tishbores" to refer to the calculation of area.

Ga'aly Mesechos

The Ga'aly Mesechos agrees with the explanation of the Me'iri and explains the language of the Rabbis of Caesarea to always base themselves on the square and calculate the area of the circle based on the area of the square. The area of a square is easy to calculate (side times side) while the area of a circle is harder as it involves π, which is inexact. Thus when the Rabbis of Caesarea said "*igula migo ribu'a **rivah***" (a circle that is inscribed inside a square, **one-quarter**) they mean line 19; the area of a circle circumscribed by a square is the area of the square **minus one-quarter** the area of the square (24 = 32 – 32/**4**). And when they said "*ribu'a migo igula **palga***" (a square that is

[12] [Me'iri 1], the other versions are similar and vary from each other only in minor textual differences. [Me'iri 1] has the same conclusions but presents it in a different order and in more detail.

[13] This can be seen by considering the square AEBO and the definition of diagonal above as side (7/2) times the square root of 2. Thus 7/2*√2 = 7/√2 = 4.95. Or it can directly calculated by the Pythagorean theorem as:
AB = $\sqrt{(AE)^2 + (EB)^2} = \sqrt{(7/2)^2 + (7/2)^2} = \sqrt{2*(7/2)^2} = \sqrt{2*(7*7)/(2*2)} = \sqrt{(7*7/2)} = 7/\sqrt{2} = 4.95$.

inscribed inside a circle, **one-half**) they mean line 20; the area of a circle with an inscribed square is the sum of the area of the square **plus one-half** the area of the square (24 = 16 + 16/**2**).

Gra on Eruvin 76b

A free translation of the Gra's comment follows but I do not understand how it resolves the difficulties.

> The Gra quotes Tosafos *D"H* R' Yochanan, that R' Yochanan was in error; and says *chas v'sholom* to say he made an error. Rather, when Rabbi Yochanan says *b'hakayfoh*, he means the outside square (a word(s) was unclear in the original text) and the length of the diagonal [of the square and the diameter] of the inscribed circle is 5 + 3/5 + ¼*(1/5) [= 100/20 + 12/20 + 1/20 = 113/20 = 5.65 (line 7)]. Therefore, the perimeter of the outside square is 22 3/5 [= 22.6 (line 22), approximately = 24].
>
> As for the Gemara which says two Tefachim and a *mahshehu* [insignificant amount] below ten Tefachim, it is explained as Rashi does, along the circumference of the circle to one side (arc LB) which is greater than two by only ½ of 1/5 [= 2.1 (line 13)]. [The length of arc LB ranges from 2.1 to 2.22 depending of the approximations used for Pi and the length of the diagonal.]

Conclusion

Based on the Me'iri, who says so explicitly, and Tosafos and the Ritva, from whom we can derive it, Rabbi Yochanan is talking about area and is correct in both Succos and Eruvin. Rabbi Yochanan's statement that "the circumference of the Sukah must be large enough to seat 24 people in it" does not mean that the **circumference** must be 24 Amos, but that there must be room for 24 people occupying 24 square Amos **inside the circumference** -- in other words, the **area** of the circle must be 24 square Amos! (line 14)

The Rabbis of Ceasarea are then brought as proof and state that the area of a circle that is drawn around a square which is 4 by 4 is calculated by subtracting ¼ of the area of the circumscribing square (line 19) or adding ½ of the area of the inscribed square (line 20) and is exactly[14] equal to 24 square Amos.

This is what Rabbi Yochanan meant when he said that the circle must have within its circumference an area of 24 in both Succos and Eruvin.

Further when Rabbi Yochanan states in Eruvin that in order to get the inscribed square of 4 by 4 Tefachim below a height of 10 Tefachim, at least 2 Tefachim and a bit of the circular window must be below ten Tefachim; he is talking about the area of segment CBL which is exactly[15] 2 square tefachim (line 21). If 2 square tefachim and a bit of the circle are below 10 tefachim, the bottom of the 4 by 4 square will therefore be below 10 tefachim and the *Pesach* (opening) is valid, and allows the *Chatzeros* to be joined in one *Eruv*.

Thus if we view the statements of Rabbi Yochanan and the Rabbis of Caesarea as referring to area and with the value of π equal to 3, they are exactly correct.

[14] Assuming π equals 3.
[15] Assuming π equals 3.

Appendix A

The Gemara explained, taken from [Kornfeld] verbatim.

Succa 8a,b

1) A Round Sukah

(a) (R. Yochanan) A round Sukah is Kosher if its circumference is large enough to contain 24 people.
(b) Question: R. Yochanan's requirement aligns with no other opinion, even the largest opinion (Rabbi, who says that any Sukah which is not 4x4 Amos is Pasul)!?

1. A person can sit in a space of 1 Amah (thus R. Yochanan's 24 people must be 24 Amos).
2. But we know that the relationship of the circumference to the diameter is 3 to 1.
3. Then taking the largest requirement for the width of a Sukah, Rebbi's (four Amos), it should suffice to have a circumference of only 12 Amos (not 24)!

I Answer: This relationship between the circumference and diameter applies only to a circle.

1. The perimeter of a square, though, is larger than the ratio of 3:1.
2. Thus in order to have a circle which has a circumference equal to the perimeter of a 4 X 4 Amos square, the circle must be larger than 12 Amos in circumference].

(d) Question: But that still does not give us 24!?

1. A square is a fourth (25%) larger than a circle (Rashi: a circular Amah is surrounded by a 3 Amah circumference, while a square Amah is surrounded by a 4 Amah perimeter).
2. It should then suffice to have a circumference of only 16 Amos (and still not 24)!

(e) Answer: This formula (that a square is 25% larger than a circle) applies only to a circle which is inscribed within a square.

1. The diameter of the inscribed circle is equal to the length of a side of the square.
2. However, the size of a circle which is circumscribed around a square (i.e. the square is inside the circle, such that the diameter of the circle is equal to the diagonal of the square) must be larger, because of the arcs of the circle.

(f) Question: But that would still give us only 16 4/5 Amos!?

1. The relationship of a side of a square to its diagonal is 1 to 1 2/5.
2. Thus, the diagonal of the square, which is the same as the diameter of the circle around it, is 4 X 1 2/5 = 5 3/5.
3. Since the circumference of a circle is 3 times its diameter, the circumference of this circle is 5 3/5 X 3 = 16 4/5).
4. It should then suffice to have a circumference of only 16 4/5 Amos (and still not 24)!

(g) Answer: R. Yochanan was not exact in his number.
(h) Question: We only say that one was not exact in his number when the difference is small, not when it is so large (24 vs. 16 4/5)!?
(i) Answer (Mar Kashisha): R. Yochanan's number was based on the assumption that 3 persons fit in 2 Amos (not one person in one Amah).
(j) Question: If so, R. Yochanan was saying that the size of the circumference of the round Sukah needs to be only 16 Amos (3 person in 2 Amos = 24 persons in 16 Amos), yet we concluded that the circle around a square which is 4 X 4 Amos must have a circumference of 16 4/5 Amos!
(k) Answer: R. Yochanan was not exact in his number.
(l) Question: We only say that one was not exact in his number when he was rounding off to a figure that would be stringent.

1. Here R. Yochanan (who says that the circumference must be 16) is rounding off to a figure that would be lenient.
2. This would allow for a smaller Sukah than actually required!

(m) Answer (R. Asi): R. Yochanan was not including the place where the person sits, which makes the measured circumference 18, not 24.

1. R. Yochanan's number was indeed based on the assumption that one person fits in one Amah.
2. However, R. Yochanan was not including in the size of the Sukah the place which the sitting person occupies.
3. Rather, the Sukah is measured from *within* (or inside) the place where the person sits.
4. Hence, if 24 people sit around the Sukah and we do not include the Amah in which they sit, the circumference of the circle comes out to be 18 (Rashi: we subtract two Amos from the diameter of the circle [one Amah on each 'side' of the Sukah] leaving a diameter of 6. The circumference of such a circle is 18).

(n) Question: But we determined that it is enough to have a circumference of 16 4/5 Amos (and not 18)!
(o) Answer: That is where R. Yochanan was not exact in his number, and his inexactness was a Chumra.

2) The Geometrical Formulae Of The Rabanan D'kesari

(a) The circle that comes out from within a square is a fourth (25%) smaller than the square around it.

8b--------------------------------------8b

(b) The square that comes out from within a circle is one half (50%) smaller than the circle around it.
(c) Question: This is not so, for we see that the circle circumscribed around a square is not so much larger. (Question remains unanswered. See commentaries)

1) [line 1] **RIBU'A** – a square

2) 2) [line 2] **KAMAH MERUBA YESER AL HA'IGUL? REVI'A!** – how much is the perimeter of a square greater than the circumference of a circle (whose diameter is the same length as the side of the square)? One fourth (of the perimeter of the square which is one third of the circumference of the circle)! (See Insights; see Graphic to Eruvin 76.)

3a) [line 3] **IGULA D'NAFIK MIGO RIBU'A** – a circle that is inscribed inside a square

b) [line 4] **RIBU'A D'NAFIK MIGO IGULA** – a square that is inscribed inside a circle

3) [line 6] **MURSHA D'KARNASA** – the projections of the corners

4) 5) [line 6] **KOL AMSA B'RIBU'A, AMSA U'TREI (CHUMSHA) [CHUMSHEI] B'ALACHSONA** – for every Amah along each side of a square the diagonal increases by one and two-fifths of an Amah

6) [line 8] **SHIVSAR NAKI (CHUMSHEI) [CHUMSHA]** – 17 less a fifth, i.e. 16 4/5 The diagonal of a square with a side of four Amos is 4 X 1 2/5 = 4 8/5 = 5 3/5 Amos
The circumference of a circle with a diameter of 5 3/5 Amos is 3 X 5 3/5 = 15 9/5 = 16 4/5 Amos (using 3 as the rounded value of pi)

7) [line 13] **KAMAH HAVAH LEHU? SHITSAR!** – how much of a circumference do the 24 people make up? 16 Amos!

Sukah 8

1. **The Mathematical Formulae Of The Rabbis Of Caesarea**

Question: After analyzing the statement of Rabbi Yochanan, who said that a Sukah built in the shape of a circle must be large enough to seat 24 people around its circumference, the Gemara mentions the geometrical theorem of the Rabbis of Caesarea. The Rabbis of Caesarea said that the circumference of a circle inscribed inside of a square is 25% less than the square's perimeter, and the circumference of a circle circumscribed around the outside of a square is 50% more than the square's perimeter. Accordingly, the circumference of the circle drawn around the 16-Tefach perimeter of a square is 50% larger, or 24 (that is, take 50% of 16 and add it to 16).

The Gemara concludes (8b) that this theorem is incorrect, as one can see. We know that the actual relationship of the perimeter of an inscribed square to the circle around it, according to Chazal, is 3 * (1.4 * s), where 3 is used for pi (Eruvin 13a) and "s" equals the length of a side of the square. (The relationship between the side of a square and its diagonal – which is also the diameter of the circumscribed circle – is 1:1.4, according to Chazal). If so, the circumference of a circle circumscribed around a square with sides of 4 Tefachim is 3(1.4 * 4), or 16.8 – and not 24!

How did the Rabbis of Caesarea make such a mistake?

Answers:

(a) **Tosafos** (8b, DH Rivu'a; Eruvin 76b, DH v'Rebbi Yochanan) suggests that the Rabbis of Caesarea were not giving the relationship of the *perimeter* of the inner square to the *circle* around it. Rather, they were giving the relationship of the *area* of the inner square to an *outer square* that is drawn around the circle which encloses the inner square. This is what they meant by saying that "when a circle is drawn around the outside of a square, the outer one's (i.e., the outer *square's*) perimeter is 50% larger than the inner one's." (See the second picture printed in Tosafos.)

According to Tosafos, Rabbi Yochanan (both here and in Eruvin 86a) misunderstood the Rabbis of Caesarea.

(b) The Gemara comments that we can see that the circle around a square is not as large as the Rabbis of Caesarea posit. Based on the comments of Rashi elsewhere, though, we might suggest that Gemara is commenting only about the mathematical correctness of their statement; however, when considering the actual Halachic applications, we do take into account their formula. In fact, we find in Eruvin (76a) that Rashi seems to have no difficulty with the statements of the Rabbis of Caesarea and Rabbi Yochanan. Perhaps Rashi held that the Rabbis of Caesarea were proposing a Halachic stringency: when determining a value (such as the circumference of a circle) by using the diagonal of a square for the purpose of a practical application in Halachah, we consider the diagonal to be equal to the sum of the two sides of the square or rectangle between the ends of the diagonal (since the lines of those two sides go from one end of the diagonal to the other). The reason for this is to prevent people from confusing the diagonal and the sum of two sides. In addition, physical reality does not permit for the application of puristic mathematics (for one reason, the actual diagonal of a square is the length of the side times the square root of two, which is an irrational number; second, it is not possible to draw a perfectly exact line or angle in the physical reality), and therefore the figure given as the diagonal of a square for purposes of determining Halachic applications (such as the size of a circular Sukah around that square) must take into consideration the largest possible diagonal of the right angle, which is the sum of the two sides. (Thus, if the sides of inscribed square are each 4 Tefachim, then the diagonal is viewed to be *8* Tefachim. The circle around that square, then, must have a diameter of 8 Tefachim, which means that its circumference must be *24* Tefachim, and not 16.8 which is what it would be based on the *actual* diameter of the square.)

It could be that Rashi is consistent with his opinion elsewhere (Shabbos 85a, Eruvin 5a, 78a, 94b), where Rashi seems to count the diagonal of a rectangle as the sum of the two sides between the two ends of the diagonal. Rashi may hold that such a Halachic definition is applied and may be relied upon entirely, both as a leniency and a stringency, with regard to Rabbinic rulings. (M. Kornfeld)

I Perhaps it is possible to propose an entirely new explanation for the statement of the Rabbis of Caesarea. The Rabbis of Caesarea and Rabbi Yochanan are perfectly correct. Perhaps Rabbi Yochanan's statement that "the circumference of the Sukah must be large enough to seat 24 people in it" does not mean that the *circumference* must be 24 Amos, but that there must be 24 Amos *inside* the circumference – in other words, the *area* of the circle must be 24 square Amos!

The area of a circle that is drawn around a square which is 4 by 4 is calculated by multiplying pi by the radius squared. The radius of the circle around a square which is 4 by 4 is half of the diagonal (5.6), which is 2.8. Let us use the Halachic estimate of pi=3. Then: 3 * (2.8)(2.8) = 23.52, or 24.

This is what Rabbi Yochanan meant when he said that the circle must have within its circumference an area of 24 (he rounded up to 24 as a Chumra)! (According to this explanation, we may accept the Ritva's suggestion that the words "v'Lo Hi..." do not belong in the Gemara and were added mistakenly by the Rabanan Savora'i.) (M. Kornfeld)

(David Garber and Boaz Tzaban of Bar Ilan University, who have been printing articles on geometric themes from Chazal for a number of years, pointed out to me that the ME'IRI in Eruvin 76 suggests this solution for the Rabbi Yochanan's I there, citing it from the Ba'al ha'Me'or. It can be traced further back to a responsum of the RIF in

Temim De'im #223. An Acharon, Teshuvos GALYA MASECHES #3, offers this solution as well. Using the mathematics of Chazal to project the area of the circle based on the area of another square that is drawn *around* it (3:4 – note that the outer square is exactly double the square drawn *inside* of the circle in area), the solution for the area of the circle is *exactly* 24 Tefachim, and not just approximately, as I concluded using the equation of pi*r*r. The Me'iri uses the word "Shibur" or "Tishbores" to refer to the calculation of area.)

Eruvin 76b

2) The Mathematical Formulae Of The Rabbis Of Caesarea

Question: The Mishnah says that a window in the wall between two Chatzeros must be at least four by four Tefachim in size, and must be within the first ten Tefachim of the height of the window, in order to be considered a Pesach (opening) and allow the Chatzeros the choice of joining together with one Eruv.

What do the dimensions of the window have to be if the window is *round*? Rabbi Yochanan made a statement that if the window is round, it "must be 24 Tefachim in its circumference, and two Tefachim (plus 4 Tefachim) and a bit of the window must be under ten Tefachim in the wall, so that if a square was inscribed in the circle a part of it would be within ten Tefachim of the ground." That is, Rabbi Yochanan is asserting that a circle drawn around a square with sides of 4 Tefachim (which has a perimeter of 16 Tefachim) has a circumference of 24 Tefachim.

The Gemara concludes that Rabbi Yochanan's mathematical calculations were based on the theorem of the Rabbis of Caesarea. They said that the circumference of an circle inscribed inside of a square is 25% less than the square's perimeter, and the circumference of a circle circumscribed around the outside of a square is 50% more than the square's perimeter. Accordingly, the circumference of the circle drawn around the 16-Tefach perimeter of a square is 50% larger, or 24 (that is, take 50% of 16 and add it to 16).

As the Gemara in Sukah (8a) points out, this theorem is clearly incorrect, as can be seen with a cursory glance. The actual relationship of the perimeter of an inscribed square to the circle around it, according to Chazal, is 3 * (1.4 * s), when 3 is used for pi (Eruvin 13a) and s = the length of a side of the square. (The relationship between the side of a square and its diagonal – which is also the diameter of the circumscribed circle – is 1:1.4, according to Chazal). If so, the circumference of a circle circumscribed around a square with sides of 4 Tefachim is 3(1.4 * 4), or 16.8 – and not 24!

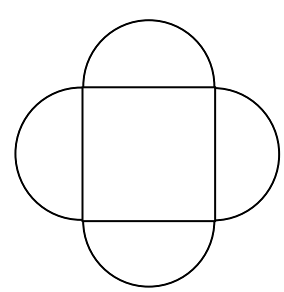
How did the Rabbis of Caesarea make such a mistake, and why did Rabbi Yochanan follow them?

Answers:

(a) **Tosafos** (DH v'Rebbi Yochanan) answers that the Rabbis of Caesarea were not giving the relationship of the *perimeter* of the inner square to the *circle* around it. Rather, they were giving the relationship of the *area* of the inner square to the *outer square* that is drawn around the circle which encloses the inner square. This is what they meant by saying that "when a circle is drawn around the outside of a square, the outer one's (i.e., the outer *square's*) perimeter is 50% larger than the inner one's." (See the picture printed in Tosafos in our Gemaras, which is slightly misleading; in the picture that appears in the **Tosafos Ha'rosh**, reproduced in our Graphics section, the inner square is shifted so that its sides are at a diagonal to the sides of the outer square. This is more demonstrative of Tosafos' point). The area of the inner square is exactly half of the area of the outer square.

According to Tosafos, Rabbi Yochanan misunderstood the Rabbis of Caesarea and made his Halachic statement regarding the relationship of the circumference of a circle to the perimeter of a square based on his misunderstanding.

(b) The **Ritva** explains that the Rabbis of Caesarea and Rabbi Yochanan are correct. When he mentioned a "round" window, Rabbi Yochanan did not mean a circular window with an imaginary square inscribed within it. Rather, he was referring to a window made in the shape of a four-leaf clover; that is, a square with four semi-circles protruding from each side (see diagram). In such a case, the perimeter of the window (i.e. the arcs of the four semi-circles) indeed add up to 50% more than the perimeter of the square around which they are drawn. In order to make sure that the square inside the clover-shaped window reaches to within a height of ten Tefachim from the ground, at least 2 Tefachim and a bit of the *radius* of the bottom semi-circle must be within ten Tefachim (since the radius of each semi-circle is 2, or half of one side of the square, which is four).



Alternatively, 2 and a bit Tefachim plus four Tefachim of the perimeter of the semi-circle must be under 10 Tefachim from the ground (as Rashi explains on bottom of 76a), since the total perimeter of each semicircle is 6 Tefachim.

I **Rashi** does not explain how to justify the formula of the Rabbis of Caesarea and how to understand Rabbi Yochanan. He seems not to have any difficulty with them. Perhaps Rashi held that the Rabbis of Caesarea were proposing a Halachic stringency: when determining a value (such as the circumference of a circle) by using the diagonal of a square, we Halachically consider the diagonal to be equal to the sum of the two sides of the square or rectangle between the ends of the diagonal (since the lines of those two sides go from one end of the diagonal to the other). The reason for this is to prevent people from confusing the diagonal and the sum of two sides. (Thus, if the sides of inscribed square are each 4 Tefachim, then the diagonal is viewed to be *8* Tefachim. The circular window around that square, then, must have a diameter of 8 Tefachim, which means that its circumference must be *24* Tefachim, and not 16.8 which is what it would be based on the *actual* diameter of the square.)

If this is why Rashi is not bothered by the formula of the Rabbis of Caesarea, then it could be that Rashi is consistent with his opinion elsewhere (Shabbos 85a, Eruvin 5a, 78a, 94b), where Rashi seems to count the diagonal of a rectangle as the sum of the two sides between the two ends of the diagonal. **Tosafos** in *all* of those places argues with Rashi, but Rashi may hold that such a Halachic definition is applied, and may be relied upon entirely, both as a leniency and a stringency, with regard to Rabbinic rulings.

(d) Perhaps it is possible to propose an entirely new explanation. The Rabbis of Caesarea and Rabbi Yochanan are perfectly correct.

Perhaps Rabbi Yochanan's statement that there "must be 24 Tefachim in its circumference," does not mean that the *circumference* must be 24 Tefachim, but that there must be 24 Tefachim *inside* the circumference – in other words, the *area* of the circle must be 24 Tefachim!

The area of a circle that is drawn around a square which is 4 by 4 is calculated by multiplying pi by the radius squared. The radius of the circle around a square which is 4 by 4 is half of the diagonal (5.6), which is 2.8. Let use the Halachic estimate of pi=3. Then: 3 * (2.8)(2.8) = 23.52, or 24.

This is what Rabbi Yochanan meant when he said that the circle must have within its circumference an area of 24 (he rounded up to 24 as a Chumra)! (According to this explanation, we may accept the Ritva's suggestion that the words "v'Lo Hi..." (in Succa) do not belong in the Gemara and were added mistakenly by the Rabanan Savora'i.) (M. Kornfeld)

(David Garber and Boaz Tzaban of Bar Ilan University, who have been printing articles on geometric themes from Chazal for a number of years, pointed out to me that the ME'IRI in Eruvin 76 suggests this solution for the Rabbi Yochanan's I there, citing it from the Ba'al ha'Me'or. It can be traced further back to a responsum of the RIF in Temim De'im #223. An Acharon, Teshuvos GALYA MASECHES #3, offers this solution as well. Using the mathematics of Chazal to project the area of the circle based on the area of another square that is drawn *around* it (3:4 – note that the outer square is exactly double the square drawn *inside* of the circle in area), the solution for the area of the circle is *exactly* 24 Tefachim, and not just approximately, as I concluded using the equation of pi*r*r. The Me'iri uses the word "Shibur" or "Tishbores" to refer to the calculation of area.)

What did Rabbi Yochanan mean that there must be 2 and a bit within a height of ten? 24 Tefachim is the area of the circle. Within that area is an inscribed square of 4 by 4, which has an area of 16 Tefachim. What is the area of the four arcs that are outside of the square? Since they are the difference between the area of circle and the square, altogether they add up to 24-16=8, and thus each one has an area of 2 Tefachim. That is exactly what Rabbi Yochanan meant when he said that in order to get the inscribed square of 4 by 4 Tefachim below a height of ten Tefachim, at least 2 Tefachim and a bit of the *area* of the circular window must be below ten Tefachim! (According to this approach, it is no longer necessary to say, as Rashi (76a) suggests, that when it says "two and a bit" it means two and a bit in addition to *four*) (M. Kornfeld)

1) [line 43] **U'SHNAYIM U'MASHEHU MEHEN B'SOCH ASARAH** – that is, a bit more than two Tefachim *besides another four Tefachim* of the circumference (that is, 6 Tefachim+ of the circumference) is within 10 Tefachim from the ground. (**RASHI**, see **SEFAS EMES**)

2) [last line] **HEKEFO** – its circumference

76b--------------------------------------76b

3) [line 1] **RIBU'A** – a square

4) [line 2] **KAMAH MERUBA YESER AL HA'IGUL? REVI'A!** – how much is the perimeter of a square greater than the circumference of a circle (whose diameter is the same length as the side of the square)? One fourth (of the perimeter of the square which is one third of the circumference of the circle)! (See Insights and Graphic.)

5a) [line 3] **IGULA D'NAFIK MIGO RIBU'A** – a circle that is inscribed inside a square

b) [line 4] **RIBU'A D'NAFIK MIGO IGULA** – a square that is inscribed inside a circle

6) [line 5] **MURSHA D'KARNASA** – the projections of the corners

7) [line 6] **KOL AMSA B'RIBU'A, AMSA U'TREI CHUMSHEI B'ALACHSONA** – for every Amah along each side of a square the diagonal increases by an one and two-fifths of an Amah

8) [line 7] **SHIVSAR NAKI CHUMSA** – 17 less a fifth, i.e. 16 4/5 The diagonal of a square with a side of four Tefachim is 4 X 1 2/5 = 4 8/5 = 5 3/5 Tefachim

The circumference of a circle with a diameter of 5 3/5 Tefachim is 3 X 5 3/5 = 15 9/5 = 16 4/5 Tefachim (using 3 as the rounded value of pi)

Appendix B

Allowable Error

The following definitions are used in the table below.

Cost – the lower number.

Selling price (sp) – the higher number.

Milgav – the "inside" ratio. – Markup – the difference as a percent of the cost (the lower number) =

(sp – cost) / cost

Milbar – the "outside ratio" – Profit – the difference as a percent of the selling price (the higher number) =

(sp – cost) / sp

There is often much confusion of *milgav*, markup and *milbar*, profit. It usually stems from thinking of a percent of, but leaving off a percent of what.

For example A, an item cost 6 and sells for 8 or example B, an item cost 6 and sells for 9.

Its *milgav*, markup[16] is A: 8 – 6 = 2, as a percent of 6 is 2/6 ~= 33.33% or one-third (1/3 ~= .3333).
B: 9 – 6 = 3, as a percent of 6 is 3/6 = 50% or one-half (1/2 = .5).

Its *milbar*, profit[17] is A: 8 – 6 = 2, as a percent of 8 is 2/8 = 25% or one-quarter (1/4 = .25).
B: 9 – 6 = 3, as a percent of 9 is 3/9 ~= 33.33% or one-third (1/3 ~= .3333).

Thus for a cost of 6 plus 1/3, the selling price could be 8 (*milgav*) or 9 (*milbar*).

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[16] If you know the cost and markup, you calculate the selling price (sp) as:
(1) sp = cost + cost * markup — basic definition
(2) sp = cost * (1 + markup) — factor out cost
For our examples A: sp = 6 * (1 + 1/3) = 6 * 4 / 3 = 8 — or 6 * (1 +.3333) = 6 * 1.3333 = 8.00
B: sp = 6 * (1 + 1/2) = 6 * 3 / 2 = 9 — or 6 * (1 +.5) = 6 * 1.5 = 9

[17] If you know the cost and desired profit percent, you calculate the selling price (sp) as:
(3) sp – sp * profit = cost — basic definition
(4) sp (1 – profit) = cost — factor out sp
(5) sp = cost / (1 – profit) — divide both sides by (1 – profit)
For our examples A: sp = 6 / (1 – 1/4) = 6 / (3/4) = 6 / 3 * 4 = 8 — or = 6 / (1 – .25) = 6 / .75 = 8
B: sp = 6 / (1 – 1/3) = 6 / (2/3) = 6 / 2 * 3 = 9 — or = 6 / (1 – .3333) = 6 / .6667 = 9.00

A related calculation is determining the markup required to generate a given profit percent. This can be derived as:
(6) cost * (1 + markup) = cost / (1 – profit) — from above, both = sp [see (2) & (5)]
(7) (1 + markup) = 1 / (1 – profit) — divide both sides by cost
(8) markup = {1 / (1 – profit)} – 1 — subtract 1 from both sides
For our examples A: markup = {1 / (1 – 1/4)} –1 = 1 / (3/4) – 1 = (1 / 3 * 4) – 1 = 4/3 – 1 = 1/3 = 33.33%
or = {1 / (1 – .25)} –1 = 1 / .75 – 1 = 1.3333 – 1 = .3333 = 33.33%
B: markup = {1 / (1 – 1/3)} –1 = 1 / (2/3) – 1 = (1 / 2 * 3) – 1 = 3/2 – 1 = 1/2 = 50%
or = {1 / (1 – .3333)} –1 = 1 / .6667 – 1 = 1.50 – 1 = .50 = 50%

A related calculation is determining the profit generated by a given markup percent. This can be derived as:
(9) (1 – profit) = 1 / (1 + markup) — from (7) above, cross multiply
(10) – profit = {1 / (1 + markup)} – 1 — subtract 1 from both sides
(11) profit = 1 – {1 / (1 + markup)} — multiply both sides by – 1
For our examples A: profit = 1 – {1 / (1 + 1/3)} = 1 – 1 / (4/3) = 1 – (1 / 4 * 3) = 1 – 3/4 = 1/4 = 25%
or = 1 – {1 / (1 + .3333)} = 1 – 1 / 1.3333 = 1 – .75 = .25 = 25%
B: profit = 1 – {1 / (1 + 1/2)} = 1 – 1 / (3/2) = 1 – (1 / 3 * 2) = 1 – 2/3 = 1/3 = 33.33%
or = 1 – {1 / (1 + .5)} = 1 – 1 / 1.5 = 1 – .6666 = .3333 = 33.33%

The following table shows the difference associated with the various calculations of the Gemara. For each circumference, the *milgav* and *milbar* ratio is calculated against the circumference used by the Gemara of 16 4/5 = 16.8 and the actual circumference of a circle that circumscribes a 4 by 4 square π * √(32) = 17.77 (line 11). In addition, for comparison, the percent differences are shown for some the common ratios used in Halacha.

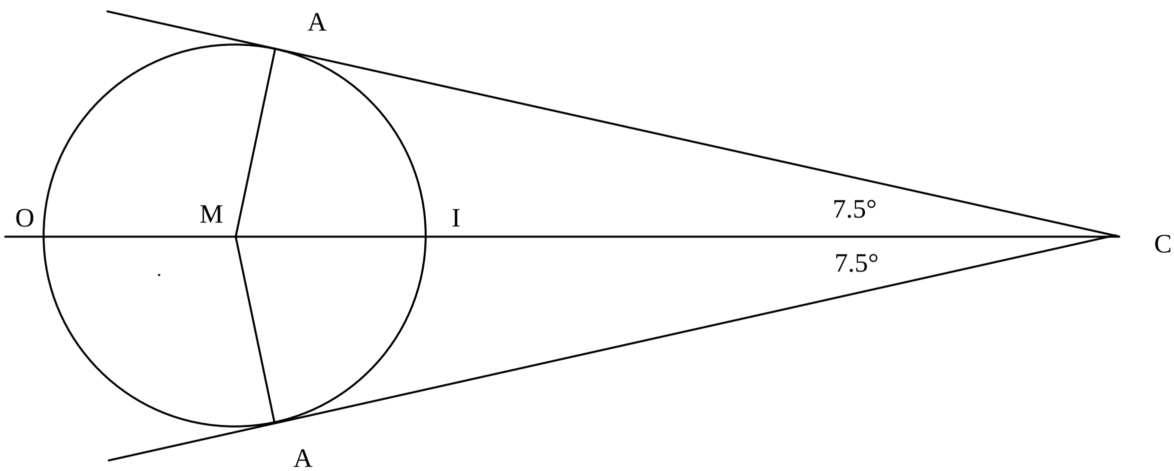
Item	Value	Talmudic value 16.8		Calculated value 17.77	
The references in ( ) refer to Appendix A		*Milgav*	*Milbar*	*Milgav*	*Milbar*
A circumference based on 16 4/5 (1.f.4)	16.8	0.00%	0.00%	5.77%	5.46%
A circumference of 24 (1.g)	24	42. 86%	30.00%	35.06%	25. 96%
A circumference based on 16 (1.j)	16	5. 00%	4.76%	11.06%	9. 96%
A circumference based on 18 (1.n)	18	7.14%	6.67%	1.29%	0.96%
The difference of Pi = 3 or π				4.72%	4.51%
The difference of Pi = π or 22/7				0.04%	0.04%
The diagonal of a square = 1 2/5 or √2 (1.f.1)	1.4			1.02%	1.01%
The difference of 22.6 and 24 [Gra]				6.19%	5.83%
A ratio of 59 to 60 [i.e. 1 out of 60]				1.69%	1.67%
Ona'ah, a ratio of 5 to 6 [i.e. 1 out of 6]				20.00%	16.67%
Trumah, a ratio of 99 to 100 [i.e. 1 out of 100]				1.01%	1.00%
Orlah & *kelai hakerm*, a ratio of 199 to 200 [1 out of 200]				0.50%	0.50%

Appendix C

Placement of 24 People

The Gemera assumes that 24 people around a circle take up 24 amos and if you consider the circle inside the people your subtract one amah from the radius. The Gemara states that a person takes up one amah space. If we consider the area of a person as a circle of diameter one amah, what would be the circumference through the center, outer edge, and inner edge of 24 such circles.

This can be solved by plain trigonometry. See the diagram below (not to scale).



To fit 24 circles around a circle, each would subtend an arc of 360° / 24 = 15° around the center (C) of the large circle.

Drawing a line from the center of the large circle (C) to the middle of one of the 24 circles (M), yields a right triangle ACM with angle C = 15° / 2 = 7.5°, angle A = 90°, and side AM = .5.

To find CM we use the rule, sine of an angle = opposite side / hypotenuse.

Sin C = AM / CM = sin 7.5° =.130526; CM = AM / sin c

Which yields:

Item	**Formula**	**Value**
AM, radius of the small circle	Diameter / 2 = 1 / 2	.5
CM, radius to the middle of the small circle	AM / sin c = .5 / .130526	3.8306
CO, radius to the outside of the small circle	CM + .5	4.3306
CI, radius to the inside of the small circle	CM - .5	3.3306
Circumference of circle with radius CM	2 * π* CM	24.07
Circumference of circle with radius CO	2 * π* CO	27.21
Circumference of circle with radius CI	2 * π* CI	20.93

For completeness, to find CA we use the rule, tangent of an angle = opposite side / adjacent side.

Tan C = AM / CA = tan 7.5° =.131652 = .5 / CA

CA = AM / tan c =.5 / tan 7.5° = .5 / .131652 = 3.7979